Last Time: Mathematical Induction.

Suppose $P(n)$ is a predicate on $\mathbb{N}$. Then

$\star$ $\forall_n P(n) \equiv P(0) \wedge \forall_k [P(k) \to P(k+1)]$.

Prop: $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$ for all $n \geq 1$.

NB: We'll prove $P(1) \wedge \forall_{k \geq 1} [P(k) \to P(k+1)]$.

This is equivalent to $\forall_{n \geq 1} P(n)$ ...

Pf: Let $P(n)$ denote the statement $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$.

We proceed by mathematical induction.

Base Case: When $n = 1$ we seek to verify $\sum_{k=1}^{1} k^2 = \frac{1}{6} 1 \cdot (2)(3)$.

On the one hand $\sum_{k=1}^{1} k^2 = 1^2 = 1$. On the other hand, $\frac{1}{6} \cdot 1 \cdot 2 \cdot 3 = \frac{1}{6} \cdot 6 = 1$, so the left and right sides of our proposed equality are indeed equal.

Hence $P(1)$ holds.

Inductive Step: Suppose $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$ for some $n \geq 1$ (i.e. $P(n)$ holds for some $n \geq 1$).

(WTS: $P(n+1)$ holds, i.e. $\sum_{k=1}^{n+1} k^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$)

We compute as follows:

$$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + (n+1)^2$$

$$= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \quad \text{(by the Inductive Hypothesis, i.e. P(n))}$$

$$= (n+1)\left(\frac{1}{6}n(2n+1) + (n+1)\right)$$

$$= (n+1)\left(\frac{1}{6}n(2n+1) + \frac{1}{6}(6(n+1))\right) \quad \text{(by } \tfrac{1}{6}\cdot 6 = 1\text{)}$$

$$= (n+1)\,\frac{1}{6}\left(n(2n+1) + 6n + 6\right)$$

$$= \frac{1}{6}(n+1)\left(\left(n(2n+1) + 2n\right) + 4n + 6\right)$$

$$= \frac{1}{6}(n+1)\left(n(2n+3) + 2(2n+3)\right)$$

$$= \frac{1}{6}(n+1)(n+2)(2n+3)$$

by basic arithmetic. Hence $\sum_{k=1}^{n+1} k^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$.

Thus P(n+1) holds, and we've shown $P(n) \to P(n+1)$.

As n was arbitrary, we have verified the inductive step.

Hence The original statement holds by math. induction. ∎

Recall: The Fibonacci numbers are defined by

initial conditions $F_1 = 1$ and $F_2 = 1$ and

recursion $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.

The first twenty Fibonacci numbers are:

n	F_n
1	1
2	1
3	$F_2 + F_1 = 2$
4	$1+2=3$
5	$2+3=5$
6	8
7	13
8	21
9	34
10	55
11	89
12	144
13	233
14	377
15	610
16	987
17	1597
18	2584
19	4181
20	6765
⋮	⋮

Q: Is there a nice formula for $\sum_{k=1}^{n} F_k$?

n	1	2	3	4	5	6	7	8
F_n	1	1	2	3	5	8	13	21
$\sum_{k=1}^{n} F_k$	1	2	4	7	12	20		

Conjecture: we think $\sum_{k=1}^{n} F_k = F_{n+2} - 1$ based on our table...

Q: How could we prove this? A: Induction!

$\forall_{n \geq 1}[P(n)]$

Prop: For all $n \geq 1$ we have $\sum_{k=1}^{n} F_k = F_{n+2} - 1$.

pf: Let $P(n)$ denote the statement $\sum_{k=1}^{n} F_k = F_{n+2} - 1$ for all natural numbers $n \geq 1$. We proceed by induction on n.

$P(1) \wedge \forall_{n \geq 1}[P(n) \to P(n+1)]$

Base Case: (WTS: $P(1)$) When $n = 1$, we have

$$\sum_{k=1}^{n} F_k = \sum_{k=1}^{1} F_k = F_1 = 1 \quad \text{and} \quad F_{n+2} - 1 = F_3 - 1 = 2 - 1 = 1.$$

Hence $\sum_{k=1}^{n} F_k = 1 = F_{n+2} - 1$ for $n=1$, so the base case holds.

$\forall n \geq 1 \, [P(n) \to P(n+1)]$

Inductive Step: Assume $\sum_{k=1}^{n} F_k = F_{n+2} - 1$ for some $n \geq 1$.

(i.e. we assumed $P(n)$ holds, so WTS: $P(n+1)$ holds).

Now we compute as follows:

$$\sum_{k=1}^{n+1} F_k = \left(\sum_{k=1}^{n} F_k\right) + F_{n+1} \qquad \text{(peel off last term)}$$

$$= (F_{n+2} - 1) + F_{n+1} \qquad \text{(by inductive hypothesis)}$$

$$= (F_{n+2} + F_{n+1}) - 1 \qquad \text{(by basic arithmetic)}$$

$$= F_{n+3} - 1 \qquad \text{(by Fibonacci recursion)}$$

$$= F_{(n+1)+2} - 1 \qquad \text{(by basic arithmetic)}$$

Hence $\left[\sum_{k=1}^{n+1} F_k = F_{(n+1)+2} - 1, \text{ so } P(n+1) \text{ holds}\right]$

As n was arbitrary, we see the inductive step holds.

Hence the original statement holds by induction. ☒

Prop: For all natural numbers $n \geq 1$ we have

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}.$$

pf: Let $P(n)$ denote the statement $F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$ for all natural numbers $n \geq 1$. We proceed by induction to show $P(n)$ holds for all $n \geq 1$.

Base Case: When $n = 1$ we have

$$F_n F_{n+2} - F_{n+1}^2 = F_1 F_{1+2} - F_{1+1}^2 = F_1 F_3 - F_2^2 = 1 \cdot 2 - 1^2 = 1 \quad \text{and}$$

$$(-1)^{n+1} = (-1)^{1+1} = (-1)^2 = 1. \quad \text{Hence } F_n F_{n+2} - F_{n+1}^2 = 1 = (-1)^{n+1}$$

in this case, so the base case holds.

Inductive Step: Assume $F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$ for some natural number $n \geq 1$. (WTS: $P(n+1)$ holds). We compute

$$\begin{aligned}
\left[F_{n+1} F_{(n+1)+2} - F_{(n+1)+1}^2\right] &= F_{n+1} F_{n+3} - F_{n+2}^2 \\
&= F_{n+1}(F_{n+2} + F_{n+1}) - F_{n+2}^2 \\
&= F_{n+1} F_{n+2} + F_{n+1}^2 - F_{n+2}(F_{n+1} + F_n) \\
&= F_{n+1} F_{n+2} + F_{n+1}^2 - F_{n+2} F_{n+1} - F_n F_{n+2} \\
&= F_{n+1}^2 - F_n F_{n+2} = (-1)(F_n F_{n+2} - F_{n+1}^2) \\
&\overset{\text{IH}}{=} (-1)(-1)^{n+1} = \left[(-1)^{(n+1)+1}\right]
\end{aligned}$$

Thus $P(n+1)$ holds, so the induction step holds. We conclude the original statement by mathematical Induction. ▨

Defn: The factorial of n is defined recursively by $0! := 1$ and $(n+1)! := n!\,(n+1)$ for all $n \geq 0$.

Prop: For all $n \geq 4$ we have $2^n < n!$ pf: Delayed... ▨